DORMANT MISSILE INERTIAL GUIDANCE AND CONTROL SYSTEM PHASE I STUDY PROGRAM

Supplemental Task 4

Rotating Pendulum Accelerometer

Prepared by

SPERRY GYROSCOPE COMPANY
DIVISION OF SPERRY RAND CORPORATION
GREAT NECK NEW YORK

Contract No. AF 04(694)-346

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Prepared for

HEADQUARTERS
BALLISTIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
NORTON AIR FORCE BASE, CALIFORNIA

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ABSTRACT

This report documents the results of a test program on a Rotating Pendulum Accelerometer (RPA) to determine its applicability to the Dormant Missile Inertial Guidance and Control System.

The RPA has attracted attention in this program because its accuracy is independent of elastic properties of matter, critical geometry, and slowly varying parasitic forces. Accuracy is dependent only upon the repeatability of an angular null indication, and the ability to measure time.

Tests were performed primarily to determine stability and quick starting characteristics. The feasibility model originally fabricated was modified to attain the degree of quality expected of future models. The tests demonstrated feasibility, and performed in the manner predicted, showing the following characteristics:

- Low noise
- Stability required by dormant system
- Ability to start quickly
- Capability of perfect temperature compensation

On the basis of these conclusions, it is strongly recommended that the following program be authorized:

- Design and fabrication of an engineering prototype RPA (preferably with a synchronous drive to simplify computation, etc.)
- Functional and environmental test of the engineering prototype
- Study of the possibility of modifications to the acceleration sensor, auxiliary equipment, and data processing methods. It is intended that these modifications would lead to simplification of equipment, improved performance and accuracy, and shorter reaction time.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS*

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*Illustrations appear immediately after the section in which they are referenced.

SECTION 1

INTRODUCTION

This report presents the results of a test program to determine the applicability of the Rotating Pendulum Accelerometer (RPA) to the requirements of the Dormant Missile Intertial Guidance and Control System. The RPA is particularly attractive for the dormant missile application because its accuracy is independent of the elastic properties of matter, critical geometry, or slowly varying parasitic forces. The novel approach of the RPA makes its fundamental accuracy dependent only upon the repeatability of an angular null indication and the ability to measure time.

1.1 BACKGROUND

The RPA was conceived by Ford Instrument Co. in 1958 and was patented (No. 2, 936,624) in May 1960. Initial interest was retarded to some extent because of the novelty of the approach and its dependency upon high digital counting rates. However, over the past few years, the state-of-the-art of digital circuitry has advanced to the point where circuitry capable of counting at the required rates is readily available. This has resulted in renewed interest in the RPA. In May 1962, Ford Instrument Co. initiated activity to fabricate and test a feasibility model of the RPA. The results of the activity clearly demonstrated the feasibility of the RPA, and verified the analytical prediction regarding performance, within the limits of accuracy of the test equipment.

1.2 OBJECTIVE AND SCOPE

The objective of the subject RPA test program was to evaluate the application of the RPA to the dormant missile requirements. This was to be accomplished by performing tests on the feasibility model to investigate its stability and quick start characteristics. Inasmuch as the RPA feasibility model was not intended to demonstrate the accuracy potential of the RPA, the accuracy of the feasibility model cannot be considered as representative of what can be obtained in a prototype unit. However, in order to investigate adequately the stability and quick start characteristics of the RPA, several modifications were made (at no cost to the program) to both the feasibility model and the associated test equipment. In addition, an analysis of several possible sources of error was performed (at no cost to the program) and is reported in Section 5.

1-1

SECTION 2
BASIC CONSIDERATIONS

The basic considerations involved in the application of the RPA are discussed in the following paragraphs.

2.1 PRINCIPLE OF OPERATION

The Rotating Pendulum Accelerometer represents a radical departure from the conventional approach to the measurement of acceleration. Whereas all known existing accelerometers involve a balance of forces to constrain the seismic mass from moving, the RPA approach allows an applied acceleration to influence the motion of a seismic mass, and computes acceleration from measurements of the motion. The seismic mass used in the RPA is an unbalanced cylinder rotating on a hydrostatic gas bearing which represents a close approximation to an ideal frictionless bearing. The effect of an acceleration-normal to the axis of rotation is to modulate the angular velocity of the rotating cylinder. This modulation is detected in the RPA by measuring the time required to rotate through successive quarter cycles of rotation.

2.2 PERFORMANCE EQUATION

It may be shown that if the applied acceleration is constant over a cycle of rotation, then a measurement of the time required to traverse quarter cycles is sufficient to determine the components of acceleration along two reference axes normal to the axis of rotation. To a first approximation, the RPA behaves like a linear device. Thus, the acceleration components may be computed from the following equation.

$$A_x = K \frac{\Delta t}{\tau^3}$$

where A_x = Acceleration component

K = Constant = $\dfrac{2\pi^3 g_E}{\omega_p^2}$

ω_p = The natural radian frequency of the RPA when oscillating through small angles in the reference gravitation field g_E

Δt = Difference in time required to traverse successive half cycles of rotation.

τ = Period of rotation.

The error involved in the use of this equation depends upon the magnitude of the modulation. Using $\Delta t/\tau$ as a modulation index, the equation has an error of approximately 1 part in 10^3 when $\Delta t/\tau$ is less than 0.03. More exact equations yielding any desired accuracy may be derived by adding higher order terms to the linear approximation. Based upon an analytic investigation of the undamped case, an expression for acceleration having errors of about 3 parts in 10^6 has the following form:

$$A_x = K_1 \frac{\Delta t}{\tau^3} + \frac{K_2}{\omega_0^4} \left(\frac{\Delta t}{\tau^3} \right)^3$$

where

$$\omega_0 = \left(\frac{1}{\tau} \right) (2\pi + 1.01 \, \pi^2 \, \frac{\Delta t}{\tau})$$

Although this equation was derived from consideration of the undamped motion of the RPA, it is expected that a solution of the same form is equally applicable to the damped case, since the magnitude of the damping in a hydrostatic gas bearing (as shown in paragraph 5.2) has a very small effect upon the magnitude and phase of the modulation.

2.3 ACCURACY

The results of an error analysis of the RPA indicate that the main sources of error are parasitic torques which vary at a rate comparable to the rate of rotation, quantization errors due to measurement of finite increments in time, and the stability of the natural frequency of the pendulum and of the clock used to measure time. Slow variations in the torque used to sustain rotation, changes in viscosity of the gas due to temperature changes, and changes in damping due to changes in bearing eccentricity under load (see paragraph 5.2) cause negligible error in the computed acceleration. In a temperature-compensated RPA, it is expected that changes in temperature will also cause negligible error (see paragraph 5.1).

Since the RPA yields a computation of acceleration or a velocity increment for each cycle of rotation, the rapidly varying parasitic torques and quantization errors cause what may be considered noise, i.e., random variations in the velocity increment about its true value. Instability in the pendulum natural frequency or clock causes changes in scale factor. The major source of error in a single velocity increment is noise. However, when a large velocity increment is measured by adding many small increments, the percentage error due to noise varies inversely with the square root of the number of increments, due to the partial cancellation of error that occurs when

increments having positive errors are added to increments having negative errors. After a very large number of increments, the error is determined only by the stability of the pendulum natural frequency and clock. It may be shown that, for equal increments, the total error is given by:

$$\frac{(\varepsilon V)rms}{V} = \sqrt{\frac{1}{n}\left[\frac{(\varepsilon \Delta V)rms}{\Delta V}\right]^2 + K^2}$$

where

$\dfrac{(\varepsilon V)rms}{V}$ = Normalized rms error in velocity

n = Number of velocity increments

$\dfrac{(\varepsilon \Delta V)rms}{\Delta V}$ = Normalized rms error in one velocity increment due to noise

K = Scale factor error.

A plot of this equation for a ten-to-one ratio of noise error to scale factor error per increment, for equal increments, is shown in figure 2-1. Based upon the test results to date and state-of-the-art of digital circuitry, it is estimated that the normalized noise would be about 5×10^{-5}, assuming the RPA rotates at about 25 cps. This figure is sufficiently low to make stability the determining factor with regard to accuracy after a large number of velocity increments.
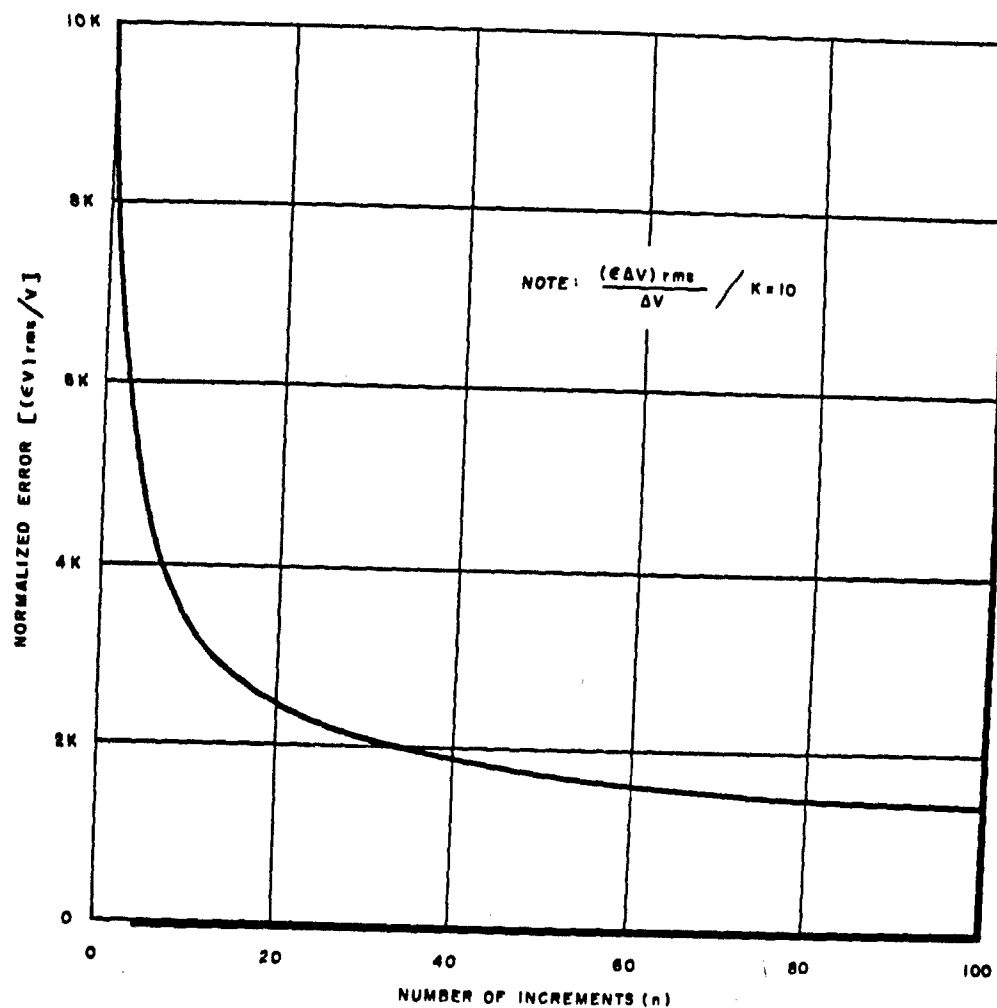
Figure 2-1. Normalized Velocity Error versus
Number of Increments

SECTION 3
DESCRIPTION OF TEST SET-UP

3.1   PARAMETERS OF RPA

The parameters of the RPA used in this test program were as follows:

- Natural frequency: 1 cps
- Pendulosity: 67 gm-cm
- Inertia: 1725 gm-cm^2
- Air bearing time constant: 163 sec.

3.2   DESCRIPTION OF TEST EQUIPMENT

The test equipment used in the performance of these tests was considered adequate to establish feasibility of the RPA for the dormant application. Figures 3-1 and 3-2 are a sketch and a photograph, respectively, of the RPA. Figure 3-3 shows a photograph of the test set-up. A brief description of the test equipment follows.

A stable base was selected on which a sine bar was placed. The accelerometer was then securely fastened to the sine bar, which permitted the RPA to be accurately positioned with respect to the vertical, through the use of calibrated gage blocks. The half cycle outputs of the accelerometer were then inserted into two Hewlett-Packard time interval counters (accurate to 1 part in 10^6), which measured the time required to traverse consecutive half cycles. These data were then recorded by digital recorders, and later processed and analyzed.

PRESSURIZED AIR

EDDY CURRENT TORQUER

UNBALANCED MASS

PRECISION GAUGE BLOCK
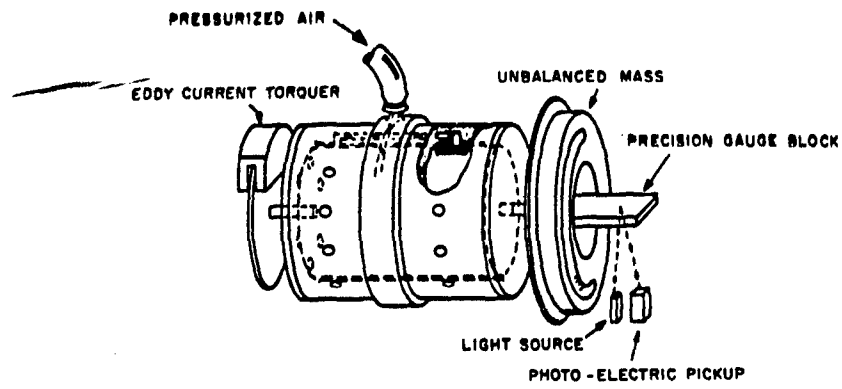
LIGHT SOURCE

PHOTO-ELECTRIC PICKUP

Figure 3-1. Rotating Pendulum Accelerometer
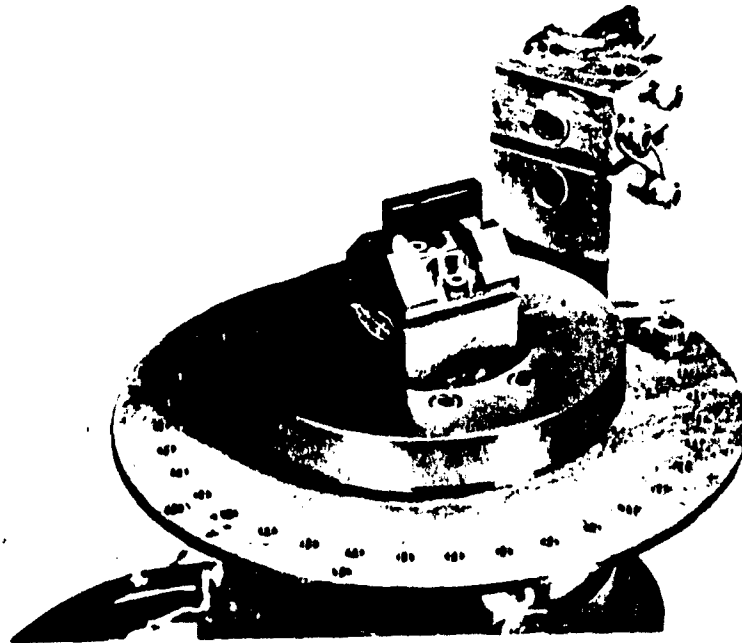(Feasibility Model), Line Drawing

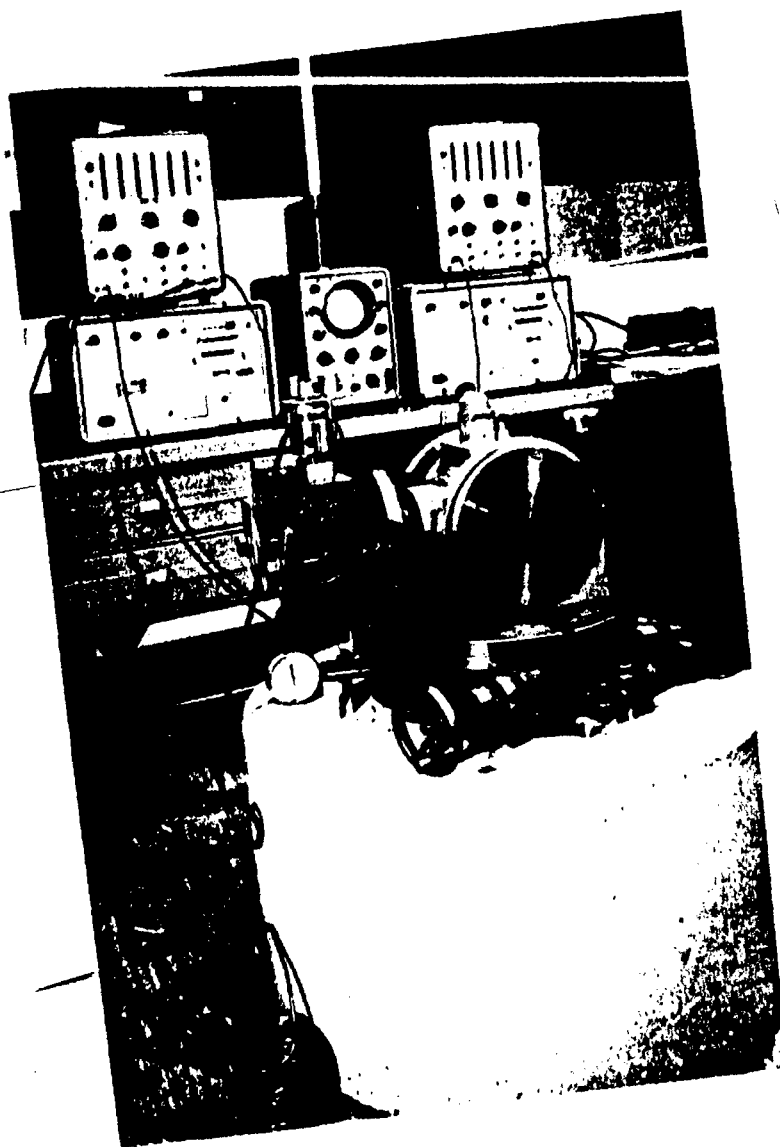Figure 3-2. Rotating Pendulum Accelerometer
(Feasibility Model)

Figure 3-3. Test Set-up for HPA

SECTION 4
TEST RESULTS

The laboratory test program conducted using the previously described model of the RPA and its associated test set-up verified the excellent stability and quick start characteristics of the RPA within the limitations of the test set-up. Due to certain inadequacies in the physical test set-up and resolution errors in the clock reference, it was impossible to demonstrate performance with the desired accuracy. Physical variations (to be described in paragraph 4.3) caused fluctuations in input acceleration that were not anticipated, while the clock resolution error caused an error in Δt, varying between plus and minus 2 microseconds. Since the full-scale value of Δt (for 0.03 modulation and $\tau = 0.4$ sec) was 12,000 microseconds, the error limits in Δt (and, effectively, in $\Delta t/\tau^3$) were $\pm 1.7 \times 10^{-4}$ due to clock resolution alone. Averaging of successive measurements of $\Delta t/\tau^3$ for constant input acceleration should reduce this error effect. A statistical data analysis was performed on measured data at two RPA speeds, and is described in paragraph 4.1.

Additional tests, described in this section, include calibration runs, linearity tests, quick start test, and stability test. The calibration runs were performed in order to determine the table bias error, the bias error due to RPA half cycle unbalance, and the linear coefficient of the RPA. The linearity run was to be used to determine the coefficients of higher order terms in ($\Delta t/\tau^3$) to correct the errors in the linear approximation. Unfortunately, lack of time made it impossible to include these data in this report. However, these data will be forwarded as soon as they become available. The quick start test demonstrated how rapidly the RPA output settles out to its final value after starting from a dormant mode. The stability test demonstrated the repeatability of the RPA for a constant input over an extended period of time.

4.1 PROBABILITY DISTRIBUTION OF NOISE

As discussed in paragraph 2.3, the RPA yields a computation of acceleration or a velocity increment for each cycle of rotation. Each of the velocity increments will have random noise superimposed upon the true value generated by rapidly varying parasitic torques, and by the quantization of time due to the measurement of time in finite increments (one microsecond in the test set-up).

The magnitude of the noise was measured at rotation rates of 1 cps and 2.5 cps. Based on 100 consecutive measurements, the probability frequency function was plotted with normalized (with respect to the 1σ value) deviation from the mean as abscissa. The results are shown in figures 4-1 and 4-2. As may be seen by comparing the measured distribution with theoretical values for the normal distribution, the noise is very nearly normally distributed. The normalized standard deviation from the mean was 0.74×10^{-4} at 1 cps and 0.72×10^{-4} cps at 2.5 cps.

The maximum error in Δt due to quantization of time is 2α, where α is the reciprocal of the clock frequency. Its standard deviation is $2\alpha/\sqrt{3}$; hence, the normalized standard deviation due to quantization of time is:

$$\sigma_{NC} = \frac{\sigma_C}{(\Delta t)max} = \frac{2\alpha/\sqrt{3}}{0.03\,\tau} = \frac{2\alpha}{0.03\sqrt{3}\,\tau}$$

where

σ_{NC} = Normalized standard deviation due to quantization

α = Reciprocal at clock frequency

τ = Pendulum period of rotation

Substitution of numbers into the equation yields

$\sigma_{NC} = 1.5 \times 10^{-4}$ at 2.5 cps

$\sigma_{NC} = 0.38 \times 10^{-4}$ at 1 cps

It may be seen that the theoretical noise due to quantization is of the same order of magnitude as the measured noise. Theoretically the noise due to random torques in the pendulum should decrease at the higher speeds (as shown in paragraph 5.3) while the noise due to quantization should increase. At some speed they will be approximately equal. Based upon the measurements made, it appears that this speed was in the range of 1 to 2.5 cps for the feasibility model and the associated test set-up. The magnitude of the noise could be reduced by increasing both the speed of rotation and the clock frequency. However, due to the cancellation of noise errors when a large number of velocity increments are summed, the noise level in the feasibility model is almost adequate for the dormant missile application. Furthermore, it is reasonable to expect that it could be substantially reduced as a result of a development program.

4.2 CALIBRATION RUNS

The purpose of the calibration runs was to determine:

- - The scale factor (of the RPA linear term).

- The table bias (error in knowledge of the vertical).

- The RPA half cycle bias (measurement error due to the difference between 180-degree half angles of the RPA due to pick off error).

The data were taken for very low modulation ($\Delta t/\tau < 0.003$), in order to stay within the highly linear region of the RPA.

In this same region of very small angular deviations from the vertical, $\sin \theta$ may be replaced by θ, and the normalized acceleration a/g is exactly equal to θ, as measured from the true vertical. Also, it may be shown that the half cycle bias causes an error in the measured value of $\Delta t/\tau^3$ equal to K_b/τ^2, where K_b is a constant equal to δ/π, δ being the radian angle difference from 180 degrees for each half cycle rotation. Thus, the measured value of $\Delta t/\tau^3$ may be expressed as

$$\frac{\Delta t}{\tau^3} = m\theta + b$$

where

$$b = \frac{K_b}{\tau^2} - m\theta_o$$

and θ_o is the table offset bias (error in knowledge of vertical). The combined bias term b will be constant if τ is constant for all data points. If values of m and b are determined for two independent calibration runs at different τ values, a pair of simultaneous equations may be solved to find values for K_b and θ_o.

Calibration runs were taken for $\tau = 0.4$ second (2.5 rps) and $\tau = 1.0$ second (1 rps). These curves were fitted by a least square error linear curve-fitting technique to determine m and b for each. The point-by-point deviation of each data value from the linear fit was normalized with respect to full scale (output for 0.03 modulation). The calibration curve and the corresponding normalized error curve at 2.5 rps are presented in figure 4-3. Each data point actually consists of the average of 10 consecutive measurements with the same θ value. At 2.5 rps, the rms deviation from the curve fit was $\sigma = 1.15 \times 10^{-4}$. These deviations are due mainly to clock resolution errors and also other test uncertainties which will be discussed in paragraph 4.3.

The zero angle reference for each of the calibration runs was taken as the point of minimum measured $\Delta t / \tau^3$. These reference angles were different for the two runs due to the K_b / τ^2 bias; the difference between the reference angles had to be used to modify the equations used to isolate K_b and θ_o. The value determined for K_b (6.106 x 10^{-4}) corresponds to a half angle error of approximately 6 minutes of arc.

4.3 LINEARITY RUNS

It had been planned to take linearity data over the range of modulation $-0.03 \leq \Delta t / \tau \leq +0.03$ and to curve-fit these data using a linear term in $\Delta t / \tau^3$ and some higher order correction terms to give a/g to an accuracy of 1 part in 10^5. (The linear curve-fit is theoretically good to 1 part in 10^3 over this range.) Unfortunately, certain unanticipated uncertainties in the laboratory set-up caused dispersions in the measured data which rendered this curve-fitting impossible. The main sources of these uncertainties are the following:

- Shifts in the mounting table which caused small changes in the effective null positions.

- Play in the sine bar mounting frame.

- Play in the mounting of the RPA (detected near the end of the program)

- Clock resolution errors.

Additional sources of small variations in RPA output include minute fluctuations in torquer voltage and air bearing turbine torques.

In spite of the above error sources, the RPA did exhibit linearity, within the modulation limits specified, of 1 part in 10^3. However, the deviation from linearity did not indicate any trend that could be curve-fitted with the higher order terms. For example, the error plot indicated apparent shifts in the vertical during a linearity run.

Most of the major error sources can be corrected readily, and reasonably good linearity data should be available in a short time.

4.4 QUICK START TEST

The quick start test demonstrates the manner in which the RPA, from a cold start, rapidly closes in on its final value. The test was run with a constant acceleration input.

The RPA was energized, spun up to its nominal rotation rate, and allowed to stabilize. Data were recorded at intervals during the first 35 minutes following the energizing.

The results are given in figure 4-4, where the deviation from the 35-minute value of output (normalized with respect to that value of output) is plotted as a function of time. Within 4 minutes, the RPA output was within a range of deviation of 3 parts in 10^4 from its final value.

4.5 STABILITY TEST

The purpose of the stability test was to demonstrate the repeatability of the RPA output over an extended period. The RPA was rigidly mounted with a fixed input acceleration, and periodically energized. After approximately 30 minutes operation, its output was recorded.

The normalized deviation of the RPA output (plotted in figure 4-5) was found to remain within about 1.5 parts in 10^4 from its mean value.

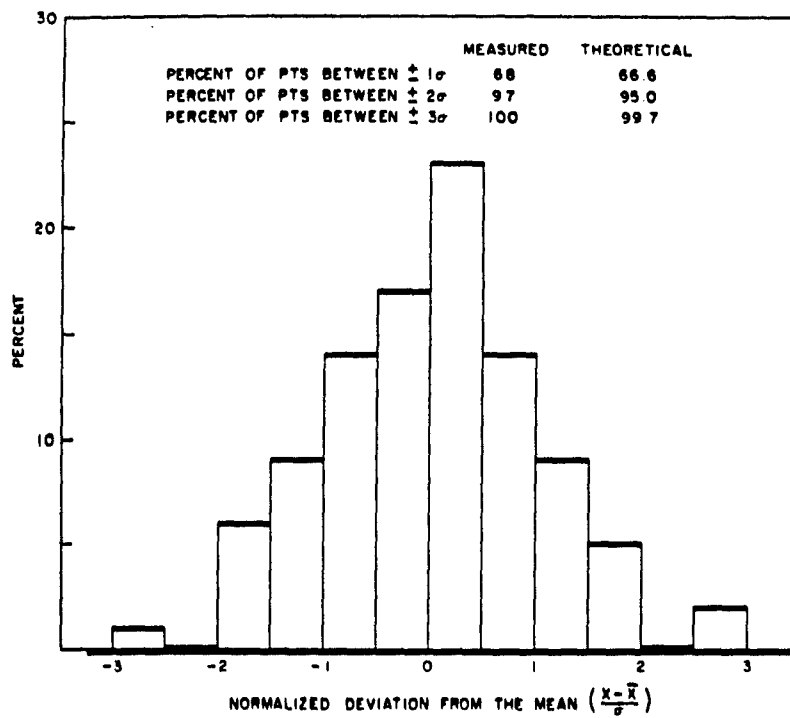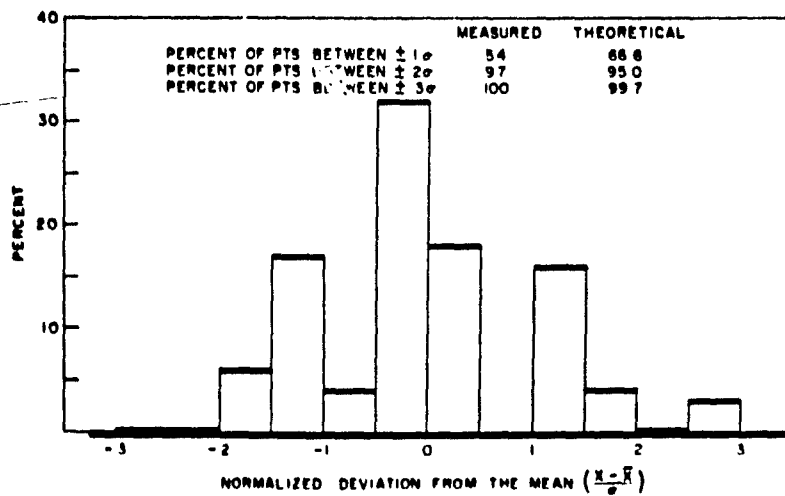Figure 4-1. Probability Frequency Function (τ = 1 Second)

Figure 4-2. Probability Frequency Function (τ = 0.4 Second)

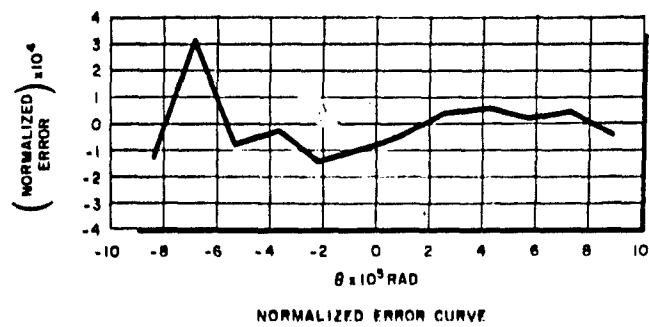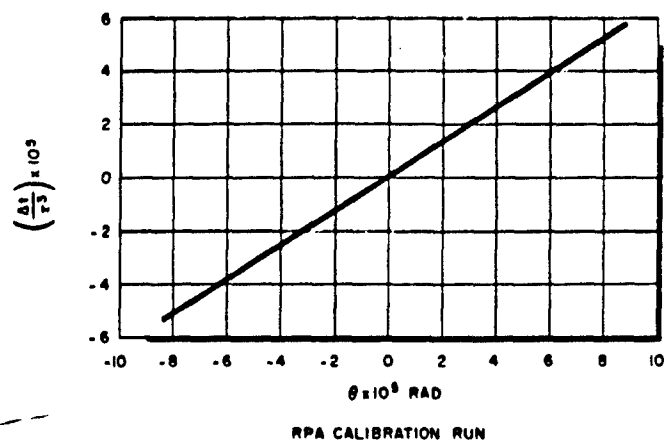RPA CALIBRATION RUN

NORMALIZED ERROR CURVE
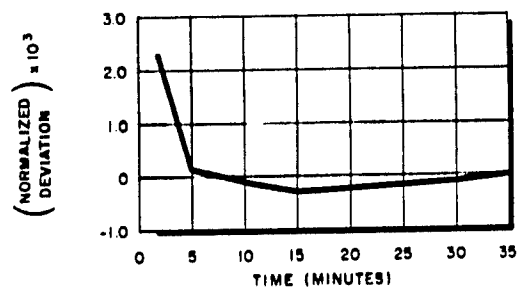
Figure 4-3. Calibration Run
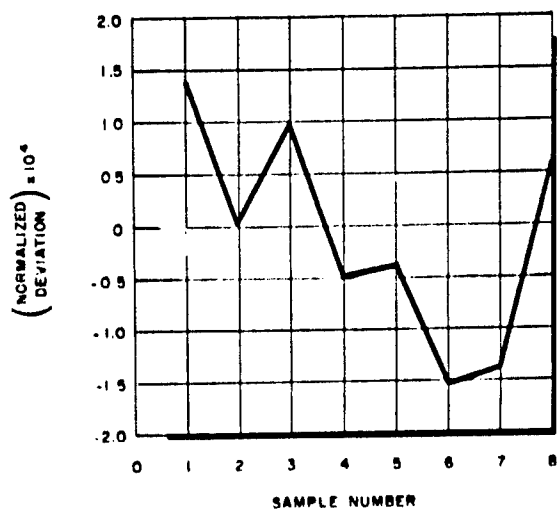
Figure 4-4. Quick Start Test

Figure 4-5. Stability Test

SECTION 5
ANALYTICAL INVESTIGATIONS

An analysis of three problem areas relating to the RPA is presented below.

5.1 TEMPERATURE STABILITY

The scale factor of the RPA varies inversely with the square of the pendulum natural frequency. The problem of temperature-compensating the RPA is therefore very similar to that of temperature-compensating a clock. In the following paragraphs it is shown that perfect compensation is feasible in a practical design.

The linear expression for the acceleration sensed by the RPA is

$$a = \frac{I \, \omega_a^3 \, \Delta t}{4 \, mr}$$

$$= \frac{I}{mr} \, 2\pi^3 \, \frac{\Delta t}{\tau^3}$$

The scale factor of the device is then proportioned to

$$\frac{I}{mr} = c$$

where I is equal to the inertia of the floated solid cylinder plus the inertia of the pendulous mass, and mr is the pendulosity of the device. Any variation in the value of $\frac{I}{mr}$ will naturally cause a corresponding variation in the computed acceleration. The largest variation in $\frac{I}{mr}$ will be caused by temperature variations; this error can be eliminated by selecting the system parameters so that they conform to the following expression:

$$\frac{v}{q} = \frac{1}{2} - \frac{mr^2}{MR^2}$$

where
- v = Coefficient of thermal expansion for the floated cylinder
- q = Coefficient of thermal expansion for the support member of the pendulous mass
- m = Pendulous mass
- r = Torque arm of the pendulous mass
- M = Mass of the floated cylinder
- R = Radius of the floated cylinder

5-1

Examination of this equation indicates that the value of $\frac{v}{q}$ must be less than $\frac{1}{2}$ since $\frac{mr^2}{MR^2}$ is a positive quantity. A simple calculation was performed in order to determine the practicality of the above solution.

Assuming a realistic value of $\frac{m}{M} = \frac{1}{10}$, and values of $v = 11 \times 10^{-6}/°C$ and $q = 24 \times 10^{-6}/°C$, which are respectively the coefficients of the thermal expansion for steel and aluminum, and $R = 1.5$ inches, a value of $r = 0.9682$ inch is obtained. These results are obviously in the area of practical design, indicating that the RPA can be designed to operate with no variation in scale factor over the temperature range. Figure 5-1 illustrates the design configuration which could be used in temperature-compensated RPA units. The pendulous mass would consist of a material with a higher coefficient of thermal expansion than the floated cylinder.

A computation was performed on the existing RPA in order to determine the variation in scale factor over the temperature range. A variation of 0.001%/°C was computed. This variation is, of course, not indicative of the scale factor sensitivity that may be expected of a temperature-compensated RPA, since the present unit was fabricated from parts already in existence.

5.2 EFFECT OF CHANGES IN DAMPING DUE TO HIGH g ENVIRONMENT

The effect of damping in the RPA gas bearing is to reduce the amplitude of modulation by a factor x and shift the phase of the modulation by an angle ζ * where x and ζ are given by:

$$x = \frac{1}{\sqrt{1 + \left(\frac{T}{2\pi\tau_p}\right)^2}}$$

$$\zeta = \tan^{-1}\left(\frac{T}{2\pi\tau_p}\right)$$

where T = Period of rotation

τ_p = Pendulum time constant

Assuming T = 163 sec

τ_p = 0.05 sec

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*"Rotating Pendulum Accelerometer", ARS Journal, April, 1961.

then substitution of τ and τ_p into the equations for x and ζ yields

$$x = 1.69 \times 10^{-9}$$

$$\zeta = 9.9 \text{ arc-seconds}$$

From the above calculations, it is evident that the effect of damping on the magnitude of acceleration is so small that even an order of magnitude increase in damping would still produce a negligible effect. However, a substantial change in damping would appreciably affect the phase shift and introduce cross coupling by the amount of the change in the phase shift. The results of an analytical investigation into this source of error due to bearing eccentricity caused by a high g environment are presented below.

It may be shown that the equation for the damping torque in an eccentric bearing is given by:

$$T_D = \frac{\mu L \omega R^3}{h} \left[\frac{8 \tan^{-1} \sqrt{\frac{1 - \delta/h}{1 + \delta/h}}}{1 - (\delta/h)^2} + \frac{4 \tan^{-1} \frac{\delta/h}{1 - (\delta/h)^2}}{1 - (\delta/h)^2} + \frac{\pi}{2} \frac{R}{L} \right]$$

where

T_D = Damping torque

μ = Viscosity of gas

ω = Angular velocity of bearing

R = Radius of bearing

h = Nominal gap

δ = Displacement of bearing from its neutral position.

The first two factors in the equation for T_D result from damping around the circumference of the bearing while the third term arises from damping on the bearing end plates.

Assuming $\frac{R}{L} = \frac{1}{2}$ (a typical value) the normalized damping torque T_{DN} is given by:

$$T_{DN} = \frac{1}{2.25\pi} \left[\frac{8 \tan^{-1} \sqrt{\frac{1 - \delta h}{1 + \delta h}}}{1 - (\delta/h)^2} + \frac{4 \tan^{-1} \sqrt{\frac{\delta/h}{1 - (\delta/h)^2}}}{1 - (\delta/h)^2} + \frac{\pi}{4} \right]$$

where T_{DN} equals the damping torque normalized with respect to the damping torque at zero eccentricity.

A plot of the percent increase in damping as a function of eccentricity is shown in figure 5-2. From the curve it may be seen that, for an eccentricity of 0.5, the change in phase shift $(\Delta \zeta)$ is:

$$\Delta \zeta = (0.13) \; 9.9 \text{ arc-seconds}$$
$$= 1.3 \text{ arc-seconds}$$

The above computation indicates the cross coupling error due to bearing eccentricity under load is acceptably small for bearing eccentricities less than 0.5.

5.3 EFFECT OF RANDOM VARIATIONS IN THE SUSTAINING TORQUE

If the torque used to sustain rotation of the RPA changes from its normal value and assumes a new constant value, the period of rotation will change. But since the period of rotation is measured and used in the computation of acceleration, there is no error. However, if the sustaining torque changes during a period of rotation, then there will be an error in the computed acceleration. The results of an analytic investigation into the errors resulting from a random variation in the sustaining torque is presented below. The analysis is based upon an investigation of the effect of the random torque on Δt, assuming no acceleration input and assuming the RPA is in a linear range of operation and the effect of damping may be neglected. It is felt that these assumptions do not introduce any significant errors into the result.

In view of the preceding assumptions, the random variations in the angle of rotation may be computed from the equation:

$$I \ddot{\theta} = M(t)$$

where

I = Moment of inertia

$\ddot{\theta}$ = Angular acceleration of rotor

$M(t)$ = Random torque

Since it has been assumed the RPA is in a linear range of operation (the modulation is small), the principle of superposition applies and the random variations in θ may be superimposed upon the constant angular velocity. The equation for Δt is approximately

$$\Delta t = \left[(\tau + \frac{\theta_\tau}{\Omega}) - (\frac{\tau}{2} + \frac{\theta_{\tau/2}}{\Omega}) \right] - \left[\frac{\tau}{2} - \frac{\theta_{\tau/2}}{\Omega} \right]$$

$$\Delta t = \frac{\theta_\tau}{\Omega} - \frac{2 \theta_{\tau/2}}{\Omega}$$

where

Δt = Difference in time between successive half cycles of rotation

τ = Period of rotation

Ω = Nominal angular velocity of RPA

θ_τ = Random variation in θ at the end of one period

$\theta_{\tau/2}$ = Random variation in θ at the end of one-half period

The above approximation solves one of the major difficulties in analytic investigations of RPA performance, specifically, the problem of inverting the solution, i.e., solving for time as a function of θ. The mean squared value of Δt is:

$$E\left[\Delta t^2\right] = E\left[\frac{1}{\Omega^2}\left(\theta_\tau - 2\theta_{\tau/2}\right)^2\right]$$

$$E\left[\Delta t^2\right] = \frac{1}{\Omega^2} E\left[\theta t^2 - 4\theta_\tau\,\theta_{\tau/2} + 4\theta_{\tau/2}^2\right]$$

where $E\left[\Delta t^2\right]$ = expected value of Δt^2

Assuming zero initial conditions, and recalling that $I\ddot\theta = M(t)$, the equation for the random variation in θ is as follows:

$$\theta = \frac{1}{I}\int_0^t d\tau \int_0^2 M(t)\,dt$$

In the process of evaluating $E\left[\Delta t^2\right]$, it is necessary to know the autocorrelation function of the random torque. To obtain a solution, an exponential autocorrelation function, which is good approximation to the autocorrelation function in a wide variety of random processes, was assumed. Thus:

$$E\left[M(t_1)\,M(t_2)\right] = \sigma^2\, e^{-\lambda|t_1 - t_2|}$$

where $E\left[M(t_1)\,M(t_2)\right]$ = Expected value of the product of the torque at times

$$t_1 \text{ and } t_2$$

σ^2 = Mean squared value of θ

λ = Constant, which determines how fast the random torque is changing

Evaluation of $E\left[\Delta t^2\right]$ in the manner previously described is a long and tedious task. The result is:

$$E\left[(\Delta t)^2\right] = \frac{\sigma^2}{(2\pi)^2 I^2}\tau^6\left[\frac{1}{6(\lambda\tau)} - \frac{2}{(\lambda\tau)^3} + \frac{6}{(\lambda\tau)^4} + \frac{2e^{-\lambda\tau}}{(\lambda\tau)^4} - \frac{8e^{-\lambda\tau/2}}{(\lambda\tau)^4}\right]$$

The normalized acceleration error is:

$$\frac{\varepsilon A}{A_m} = \frac{\frac{(\Delta t)rms}{\tau^3}}{\frac{0.03}{\tau^2}} = \frac{(\Delta t)rms}{0.03\,\tau}$$

where $\dfrac{\varepsilon A}{A_m}$ = Acceleration error normalized with respect to maximum acceleration input at $\Delta t/\tau = 0.03$

$$(\Delta t)rms = \sqrt{E\left|\Delta t^2\right|}$$

Substituting $(\Delta t)rms$ into the equation for $\dfrac{\varepsilon A}{A_m}$ yields:

$$\frac{\varepsilon A}{A_m} = \frac{\sigma}{0.03\,(2\pi)I}\;\frac{\tau^2}{4}\left[\frac{8}{2(\lambda\tau)} - \frac{32}{(\lambda\tau)^3} + \frac{96}{(\lambda\tau)^4} + \frac{32e^{-\lambda\tau}}{(\lambda\tau)^4} - \frac{128e^{-\lambda\tau/2}}{(\lambda\tau)^4}\right]^{1/2}$$

It may be noted that for a given value of $\lambda\tau$, the above equation predicts that the normalized error will vary directly with the square of the period of rotation. Hence, one would expect better performance at higher angular velocities, everything else being equal. This trend was observed in the one-σ values of the measured noise at 1 cps and 2.5 cps. However, the improvement in noise caused by parasitic torques at 2.5 cps was probably masked by the quantization noise which could account for the total magnitude of the noise at 2.5 cps (approximately 10^{-4}).

A plot of the relative noise (square root of the quantity in brackets) as a function of $(\lambda\tau)$ for constant τ is shown in figure 5-3. By expanding the exponential terms in a power series, it may be shown that a good approximation for $\lambda\tau \ll 1$ is given by:

$$\frac{\varepsilon A}{A_m} = \frac{\sigma}{0.03\,(2\pi)I}\;\frac{\tau^2}{4}\left[1 - \frac{7}{24}(\lambda\tau) + \frac{5}{96}(\lambda\tau)^2\right]^{1/2}$$

It may be noted that this equation predicts that the normalized error will reach a constant value as $(\lambda\tau)$ approaches zero, corresponding to a constant torque. This result is incorrect and is caused by the fact that Δt was computed during the first cycle of rotation after the torque was applied. The correct steady-state result should be computed by performing the required integrations over a cycle starting from t_0

and then allowing t_o to approach infinity. It is expected that the steady-state solution would show a peak at some value of ($\lambda \tau$) and approach zero as ($\lambda \tau$) approaches zero and infinity. It seems reasonable to expect the peak to occur at ($\lambda \tau$) = 1, i.e., when the noise is changing at a rate comparable to the rate of rotation. The anticipated peaking of the relative error is shown dotted in figure 5-3.
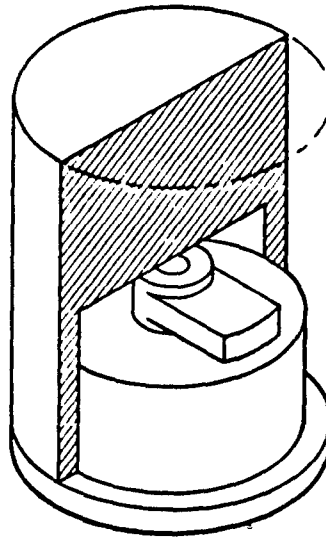
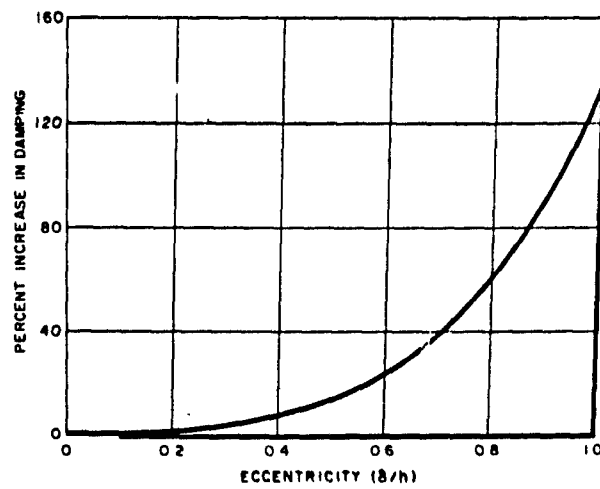Figure 5-1. Configuration of Temperature-Compensated RPA

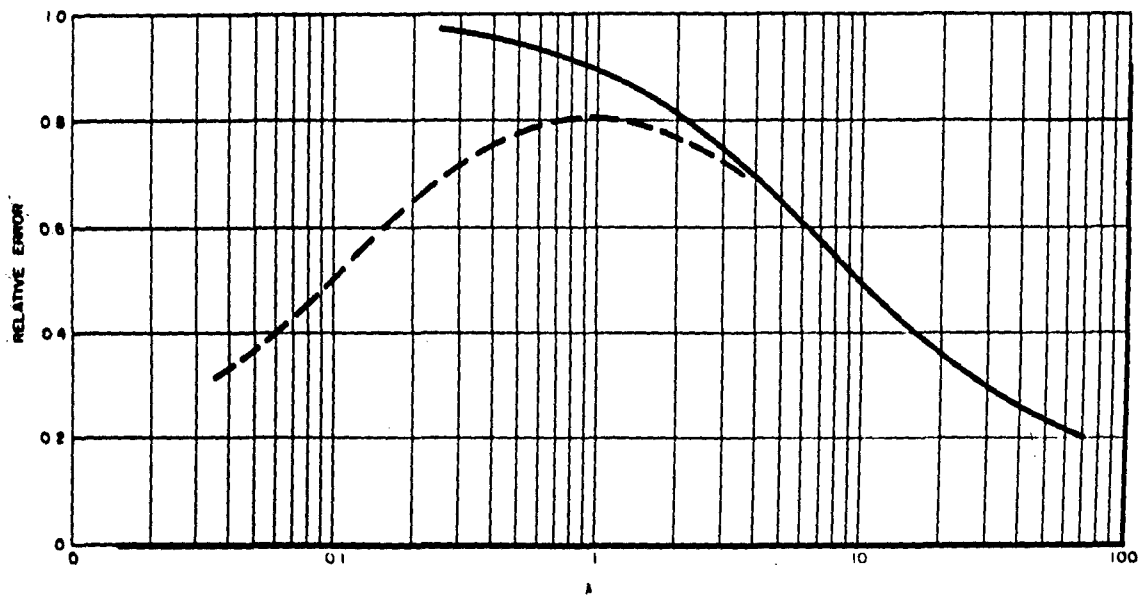Figure 5-2. Increase in Damping versus Eccentricity

Figure 5-3. Relative Error versus λ

SECTION 6
CONCLUSIONS AND RECOMMENDATIONS

6.1 CONCLUSIONS

The principal results of the RPA test program are listed below.

- The normalized (with respect to maximum acceleration input) root mean square value of noise in a single velocity increment was 0.7×10^{-4}. The noise was normally distributed about the mean and had a magnitude very nearly equal to that which results from the resolution of the 1-megacycle clock used in the test setup.

- The normalized stability error with respect to time and shutdown was approximately 1×10^{-4}.

- The quick start tests indicated that the RPA essentially reached its steady-state indication in about 5 minutes.

- An analysis of temperature sensitivity indicates that perfect temperature compensation is theoretically feasible in a practical design.

In general, the errors measured during the stability and quick start tests are considered to be more representative of the limitations of the feasibility model and its associated test equipment than of the capability of a prototype RPA. It is strongly believed that the results of the test program confirm the excellent stability and quick start characteristics predicted on the basis of an error analysis of the RPA.

6.2 RECOMMENDATIONS

Based upon the favorable results of the test program, it is recommended that a follow-on program be funded to include the tasks described in the following paragraphs.

6.2.1 Design and Fabrication of an Engineering Prototype

Inasmuch as having a variable period has a number of undesirable consequences, including a substantial computing problem, it is recommended that an engineering prototype RPA be designed and fabricated to include a synchronous driving motor connected to the rotor via a very flexible coupling which would have negligible effect upon the modulation resulting from an input acceleration. The stability and linearity of the spring constant should have negligible effect upon the accuracy of the synchronous RPA, since the torque

applied by the spring will be only sufficient to balance the damping torque. It is shown in paragraph 5.2 that substantial changes in the damping torque have negligible effect upon the accuracy of the RPA. The additional noise introduced by the hunting of the synchronous motor is not expected to affect the accuracy potential of the RPA to a significant degree.

6.2.2 Test Program

A functional and environmental test program should be conducted on the engineering model to determine stability for the dormant missile application.

6.2.3 Further Modifications

A study of further modifications to the acceleration sensor, to the auxiliary equipment, and to data processing methods should be made. These modifications would result in improved performance and accuracy, shorter reaction time or equipment simplifications. This study (intended to be performed in parallel with the design and test programs) would result in a report giving analytical data and recommendations for further modifications.